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The effects of truncation on surface-wave Hamiltonians

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The Hamiltonian form of the equations for surface waves can generate very nonlinear, realistic-looking solutions even when the Hamiltonian is truncated to low order – two or three terms – in its slope expansion. A perturbation analysis of these equations shows that most of the basic fluid behaviour is retained in the low-order terms; however, the lowest-order nonlinear equations become dramatically unstable at wavenumbers greater than g/w^2, where w is the local vertical surface velocity. One more term in the Hamiltonian mitigates this instability, extending the regime of stable slopes and wavenumbers.

1. Introduction

The traditional means of deriving solutions to the nonlinear equations of surface-wave dynamics is the Stokes expansion, which yields a perturbation-series approximation to the surface motion induced by the exact equations. Recently I and several colleagues have reported an alternative approach (West *et al.* 1987) which approximates the equations of motion in such a way that they can be implemented conveniently on a computer. This makes available solutions of arbitrarily good precision to problems of complicated or random wave motion, at the cost of some fidelity in the physical model.

This approach starts with the equations of motion in the form first used by Watson & West (1975) and expands them, or equivalently the associated Hamiltonian, in a series in surface slope. This series, truncated to finite order, defines the approximate physical system modelled by the computer. When terms up to second nonlinear order are included the usual weak resonant and non-resonant interactions are correctly modelled on formal grounds (Henyey *et al.* 1988) and, as demonstrated in West *et al.* (1987), some surprisingly realistic nonlinear motion can be generated at respectably finite slopes.

How good an approximation is the truncated Hamiltonian? To what extent are the basic fluid properties such as incompressibility and irrotationality preserved in the truncated equations? These questions are less straightforward than they sound, for in the Hamiltonian formulation the bulk fluid is not present, having been replaced by non-local operators connecting surface values of the kinetic fields – potential, velocity, and elevation rate. The exact equations are equivalent to the ordinary formulation, but in the approximate forms how are the auxiliary parameters standing for velocity to be interpreted, when their definitions have been truncated along with the Hamiltonian? I propose to approach these questions by examining the behaviour of perturbations to the flow in the form of short-wavelength wave packets, which serve as 'test particles' whose motion on the surface will provide an unambiguous measure of velocity. This effective velocity will turn out to be an orderly approximation to the actual velocity implied by the surface potential. The perturbation analysis required resembles that in Henyey *et al.* (1988) for the exact equations, and draws upon several of the results developed there.

This analysis is useful not only in clarifying the nature of the approximation but also in assessing some of the effects; for example, lingering doubts about the convergence of the Hamiltonian series in the presence of modes of wavelength shorter than the surface elevation (Brueckner & West 1988) are safely buried. However, truncation does lead to unphysical behaviour that appears first at high wavenumbers as the surface amplitudes increase. Most dramatic is an exponentially growing instability in the simplest nonlinear Hamiltonian, occurring in modes whose intrinsic phase speed is less than the local vertical velocity. In fact, it was this instability that inspired the perturbation analysis in the first place, when our early simulations of Stokes waves kept blowing up and we could find no obvious numerical explanation.

When the next term in the Hamiltonian is added this particular instability vanishes, to be replaced by another which sets in at higher wavenumbers and surface slopes.

2. The truncated canonical equations

The surface elevation ζ and potential ϕ, regarded as functions of the two-dimensional lateral coordinate $\boldsymbol{x}$ and time, obey modified forms of the kinematic and dynamic conditions,

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\zeta + w[1+(\boldsymbol{\nabla}\zeta)^2], \tag{2.1}$$

$$\frac{\partial \phi}{\partial t} = \tfrac{1}{2}\{w^2[1+(\boldsymbol{\nabla}\zeta)^2]-(\boldsymbol{\nabla}\phi)^2\} - g\zeta + \tau\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}\zeta\left(\frac{\mathrm{d}s}{\mathrm{d}\boldsymbol{x}}\right)^{-1}\right], \tag{2.2}$$

as first introduced by Watson and West (1975) to avoid convergence problems arising in the usual Stokes expansion around a mean surface; g and τ are the acceleration due to gravity and the kinematic surface tension. The equations are self-contained when the vertical velocity w is connected to ϕ by the non-local operator $\hat{\mathrm{D}}$, linear in its effect on ϕ,

$$w \equiv \hat{\mathrm{D}}\phi, \tag{2.3}$$

but depending on the surface profile ζ to all orders in such a way that $\hat{\mathrm{D}}$ reproduces the vertical derivative $\partial\Phi/\partial z$ of the bulk potential satisfying Laplace's equation and equalling ϕ on the surface, that is,

$$\phi(\boldsymbol{x}) = \Phi(\boldsymbol{x}, z)|_{z=\zeta(\boldsymbol{x})}, \tag{2.4}$$

$$\left(\boldsymbol{\nabla}^2 + \frac{\partial^2}{\partial z^2}\right)\Phi = 0. \tag{2.5}$$

Here and in what follows $\boldsymbol{\nabla}$ will stand for the two-dimensional gradient with respect to $\boldsymbol{x}$. The values of $\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla}\Phi$ at the surface are not the same, but rather

$$(\boldsymbol{\nabla}\Phi)_{z=\zeta} = \boldsymbol{\nabla}_{\mathrm{h}}\phi, \tag{2.6}$$

where

$$\boldsymbol{\nabla}_{\mathrm{h}} \equiv \boldsymbol{\nabla} - (\boldsymbol{\nabla}\zeta)\,\hat{\mathrm{D}} \tag{2.7}$$

is the operator equivalent of the horizontal gradient at constant depth. Consequently, the horizontal velocity is

$$\boldsymbol{u} = \boldsymbol{\nabla}_{\mathrm{h}}\phi = \boldsymbol{\nabla}\phi - w\boldsymbol{\nabla}\zeta. \tag{2.8}$$

The field equations (2.1), (2.2) are a canonical pair derivable from the Hamiltonian

$$H = \frac{1}{2}\int\left[\phi\hat{\mathrm{K}}\phi + g\zeta^2 + 2\tau\frac{\mathrm{d}s}{\mathrm{d}\boldsymbol{x}}\right]\mathrm{d}\boldsymbol{x} \tag{2.9}$$

via

$$\frac{\partial\zeta}{\partial t} = \frac{\delta H}{\delta\phi}, \quad \frac{\partial\phi}{\partial t} = -\frac{\delta H}{\delta\zeta}, \tag{2.10}$$

where $\hat{\mathrm{K}}$, the operator appearing in the kinetic energy term, is

$$\hat{\mathrm{K}} \equiv [1+(\nabla\zeta)^2]\,\hat{\mathrm{D}} - \nabla\zeta\cdot\nabla, \tag{2.11}$$

which when applied to ϕ produces the normal flux

$$\hat{\mathrm{K}}\phi = \frac{\partial\Phi}{\partial n}\left(\frac{\mathrm{d}s}{\mathrm{d}\boldsymbol{x}}\right) = \frac{\partial\zeta}{\partial t}. \tag{2.12}$$

See Broer (1974) and Miles (1977). Here

$$\frac{\mathrm{d}s}{\mathrm{d}\boldsymbol{x}} = [1+(\nabla\zeta)^2]^{\frac{1}{2}}, \tag{2.13}$$

and the fluid density is unity for convenience. Every term in the Hamiltonian is exact save the operator $\hat{\mathrm{D}}$, which must be developed in a series in ζ,

$$\hat{\mathrm{D}} = \sum_{n=0}^{\infty}\hat{\mathrm{D}}_n,$$

as will be described below. For moderate surface slope this is a rapidly converging series and good approximations to w can be obtained by interrupting the series at finite order m,

$$w_{(m)} = \hat{\mathrm{D}}_{(m)}\,\phi \equiv \sum_{n=0}^{m}\hat{\mathrm{D}}_n\,\phi. \tag{2.14}$$

One might consider substituting this quantity into the exact equations (2.1), (2.2) to approximate the dynamics, but the resulting system would no longer be canonical. Rather, it seems better to truncate the kinetic energy term in the Hamiltonian to a given order via

$$\hat{\mathrm{K}}_{(m)} = \hat{\mathrm{D}}_{(m)} + (\nabla\zeta)^2\,\hat{\mathrm{D}}_{(m-2)} - \nabla\zeta\cdot\nabla \tag{2.15}$$

and let the truncated Hamiltonian define the approximate field equations. For $m \geqslant 1$ this results in

$$\frac{\partial\zeta}{\partial t} + \nabla\phi\cdot\nabla\zeta = w_{(m)} + (\nabla\zeta)^2\,w_{(m-2)}, \tag{2.16}$$

$$\frac{\partial\phi}{\partial t} + g\zeta + \tfrac{1}{2}(\nabla\phi)^2 = (w^2)_{(m-1)} + (\nabla\zeta)^2\,(w^2)_{(m-3)} \tag{2.17}$$

(neglecting capillarity), where

$$(w^2)_{(m)} = \sum^{n+n'\leqslant m} w_n\,w_{n'} \tag{2.18}$$

and so on. These equations remain canonical, obeying the usual conservation laws exactly, and they retain other convenient algebraic properties that follow from the self-adjointness of the operator $\hat{\mathrm{K}}$. This property,

$$\int f\hat{\mathrm{K}}g\,\mathrm{d}\boldsymbol{x} = \int g\hat{\mathrm{K}}f\,\mathrm{d}\boldsymbol{x}, \tag{2.19}$$

follows from Green's theorem in the form

$$\int \left(f \frac{\partial G}{\partial n} - g \frac{\partial F}{\partial n} \right) \mathrm{d}s = 0 \tag{2.20}$$

applied to a pair of harmonic functions F, G, whose surface values are f and g. This property is clearly true separately at each order in ζ, that is, for each $\hat{\mathrm{K}}_n$, and also for the truncated sum $\hat{\mathrm{K}}_{(m)}$.

3. Operator expansions and properties

Suppose $F(\boldsymbol{x}, z)$ is a harmonic function whose value on the surface, $z = \zeta(\boldsymbol{x})$, is $f(\boldsymbol{x})$. For a medium of infinite depth one can write

$$F = \sum_{\boldsymbol{k}} \varphi(\boldsymbol{k}) \, \mathrm{e}^{kz + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{3.1}$$

and

$$\frac{\partial F}{\partial z} = \sum_{\boldsymbol{k}} \varphi(\boldsymbol{k}) \, k \, \mathrm{e}^{kz + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.2}$$

where k is the magnitude of $\boldsymbol{k}$. The operator $\hat{\mathrm{D}}$ is defined by

$$\hat{\mathrm{D}} f \equiv \left(\frac{\partial F}{\partial z} \right)_{z = \zeta(\boldsymbol{x})}, \tag{3.3}$$

and its specification as a functional of ζ is the central problem of the Hamiltonian formulation. Watson & West exhibited the first two terms in their original paper (1975), and subsequently Brueckner showed how to systematically construct higher-order terms in a way that is convenient for numerical work (West *et al.* 1987). Let me review this construction briefly along with an alternative one that better reveals the formal properties. First, write $f(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \zeta(\boldsymbol{x})^n \sum_{\boldsymbol{k}} k^n \varphi(\boldsymbol{k}) \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \tag{3.4}$$

by expanding the exponential $\exp[k\zeta(\boldsymbol{x})]$. If we define the scalar operator $\hat{\mathrm{k}}$ by

$$\hat{\mathrm{k}} \left\{ \sum_{\boldsymbol{k}} \varphi(\boldsymbol{k}) \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right\} \equiv \sum_{\boldsymbol{k}} k\varphi(\boldsymbol{k}) \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.5}$$

that is, as the linear operator whose effect is to multiply each Fourier component of a function by its wavenumber modulus, we then have

$$f(\boldsymbol{x}) = \hat{\mathrm{Z}} F(\boldsymbol{x}, 0) \tag{3.6}$$

with

$$\hat{\mathrm{Z}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \zeta(\boldsymbol{x})^n \, \hat{\mathrm{k}}^n. \tag{3.7}$$

The operator $\hat{\mathrm{Z}}$ is formally (though perhaps slowly) convergent when applied to band-limited physical functions F. The inverse operator produces $F(\boldsymbol{x}, 0)$ from f,

$$F(\boldsymbol{x}, 0) = \hat{\mathrm{Z}}^{-1} f(\boldsymbol{x}), \tag{3.8}$$

and because $\partial F(\boldsymbol{x}, 0)/\partial z = \hat{\mathrm{k}} F(\boldsymbol{x}, 0)$ we have

$$\begin{aligned} \hat{\mathrm{D}} f &= \hat{\mathrm{Z}} \hat{\mathrm{k}} F(\boldsymbol{x}, 0) \\ &= \hat{\mathrm{Z}} \hat{\mathrm{k}} \hat{\mathrm{Z}}^{-1} f(\boldsymbol{x}) \end{aligned}$$

or formally
$$\hat{\mathrm{D}} = \hat{\mathrm{Z}}\hat{\mathrm{k}}\hat{\mathrm{Z}}^{-1}. \tag{3.9}$$

The expansion of $\hat{\mathrm{Z}}^{-1}$ in powers of ζ can be recovered from the set of equations implied by $\hat{\mathrm{Z}}\hat{\mathrm{Z}}^{-1}-1 = 0$, examined order by order. The zeroth-order term gives $(\hat{\mathrm{Z}}^{-1})_0 = 1$, while order n gives

$$(\hat{\mathrm{Z}}^{-1})_n + \zeta\hat{\mathrm{k}}(\hat{\mathrm{Z}}^{-1})_{n-1} + \dots \frac{1}{n!}\zeta^n\hat{\mathrm{k}}^n = 0, \tag{3.10}$$

which is a recursion equation for $(\hat{\mathrm{Z}}^{-1})_n$ comprising powers of ζ, powers of $\hat{\mathrm{k}}$, and lower-order terms of $\hat{\mathrm{Z}}^{-1}$. The series for $\hat{\mathrm{D}}$,

$$\hat{\mathrm{D}} = \sum_{n=0}^{\infty} \hat{\mathrm{D}}_n, \tag{3.11}$$

is straightforwardly obtainable from (3.10) as

$$\hat{\mathrm{D}}_n = \sum_{m=0}^{n} \frac{1}{m!}\zeta^m\hat{\mathrm{k}}^{m+1}(\hat{\mathrm{Z}}^{-1})_{n-m}. \tag{3.12}$$

It is true but not obvious from this form that $\hat{\mathrm{D}}$ is independent of the location of the reference plane $z = 0$, that is, invariant to any additive constant in ζ, so that (3.11) is actually a slope expansion; see Brueckner & West (1988).

Another expansion, which reorders the terms in (3.12), makes this property explicit. A variation in surface elevation $\delta\zeta$ produces

$$\delta\hat{\mathrm{Z}} = \delta\zeta\hat{\mathrm{Z}}\hat{\mathrm{k}} \tag{3.13}$$

directly from (3.7) and consequently

$$\begin{aligned}\delta\hat{\mathrm{Z}}^{-1} &= -\hat{\mathrm{Z}}^{-1}(\delta\hat{\mathrm{Z}})\,\hat{\mathrm{Z}}^{-1}\\ &= -\hat{\mathrm{Z}}^{-1}\delta\zeta\hat{\mathrm{D}}\end{aligned} \tag{3.14}$$

(see (3.9)). Then

$$\begin{aligned}\delta\hat{\mathrm{D}} &= \delta\zeta\,\hat{\mathrm{Z}}\hat{\mathrm{k}}^2\hat{\mathrm{Z}}^{-1} - \hat{\mathrm{Z}}\hat{\mathrm{k}}\hat{\mathrm{Z}}^{-1}\delta\zeta\hat{\mathrm{D}}\\ &= \delta\zeta\hat{\mathrm{D}}^2 - \hat{\mathrm{D}}\delta\zeta\hat{\mathrm{D}},\end{aligned}$$

or
$$\delta\hat{\mathrm{D}} = [\delta\zeta, \hat{\mathrm{D}}]\,\hat{\mathrm{D}}, \tag{3.15}$$

where $[\delta\zeta, \hat{\mathrm{D}}]$ is the commutator product $\delta\zeta\hat{\mathrm{D}} - \hat{\mathrm{D}}\delta\zeta$. If $\zeta(x)$ is substituted for $\delta\zeta$ above, the nth-order term is just $\delta\hat{\mathrm{D}} = n\hat{\mathrm{D}}_n$, so that

$$\hat{\mathrm{D}}_n = \frac{1}{n}\sum_{m=0}^{n-1} [\zeta, \hat{\mathrm{D}}_m]\,\hat{\mathrm{D}}_{n-m-1}. \tag{3.16}$$

The invariance of $\hat{\mathrm{D}}_n$ to an additive constant in ζ then follows explicitly from the invariance at lower orders, starting with $\hat{\mathrm{D}}_0 = \hat{\mathrm{k}}$. The first few terms are easily constructed as

$$\hat{\mathrm{D}}_0 = \hat{\mathrm{k}}, \tag{3.17a}$$

$$\hat{\mathrm{D}}_1 = [\zeta, \hat{\mathrm{k}}]\,\hat{\mathrm{k}}, \tag{3.17b}$$

$$\hat{\mathrm{D}}_2 = \{\tfrac{1}{2}[\zeta, [\zeta, \hat{\mathrm{k}}]]\,\hat{\mathrm{k}} + [\zeta, \hat{\mathrm{k}}]^2\}\,\hat{\mathrm{k}} \tag{3.17c}$$

and so on. Applied to the field ϕ, these produce terms w_n in the series (2.14) for vertical velocity; similarly, the series for horizontal velocity is

$$\boldsymbol{u}_0 = \boldsymbol{\nabla}\phi, \quad \boldsymbol{u}_n = -w_{n-1}\boldsymbol{\nabla}\zeta. \tag{3.18}$$

When equations involving these velocities are truncated to finite order there are consequences for elementary properties such as irrotationality and incompressibility. In surface form these are stated as

$$\boldsymbol{\nabla}_{\mathrm{h}} w - \hat{\mathrm{D}}\boldsymbol{u} = 0, \tag{3.19}$$

$$\boldsymbol{\nabla}_{\mathrm{h}} \cdot \boldsymbol{u} + \hat{\mathrm{D}} w = 0, \tag{3.20}$$

and could be taken for granted, but it is instructive to prove them algebraically as follows. From (3.7) and the chain rule, along with the property $\boldsymbol{\nabla}\hat{\mathrm{k}} = \hat{\mathrm{k}}\boldsymbol{\nabla}$, we have

$$\boldsymbol{\nabla}\hat{\mathrm{Z}}F = (\boldsymbol{\nabla}\hat{\mathrm{Z}})F + \hat{\mathrm{Z}}\boldsymbol{\nabla}F, \tag{3.21}$$

and from (3.13)

$$\boldsymbol{\nabla}\hat{\mathrm{Z}}F = (\boldsymbol{\nabla}\zeta)\hat{\mathrm{Z}}\hat{\mathrm{k}}F + \hat{\mathrm{Z}}\boldsymbol{\nabla}F; \tag{3.22}$$

setting $F = \hat{\mathrm{Z}}^{-1}f$ yields

$$\hat{\mathrm{Z}}\boldsymbol{\nabla}\hat{\mathrm{Z}}^{-1} = \boldsymbol{\nabla} - (\boldsymbol{\nabla}\zeta)\hat{\mathrm{D}} = \boldsymbol{\nabla}_{\mathrm{h}}, \tag{3.23}$$

which identifies $\boldsymbol{\nabla}_{\mathrm{h}} f$ as the surface value of the ordinary horizontal gradient of F as expected. Because products of operators $\hat{\mathrm{a}}, \hat{\mathrm{b}}$ obey

$$\hat{\mathrm{Z}}(\hat{\mathrm{a}}\hat{\mathrm{b}})\hat{\mathrm{Z}}^{-1} = (\hat{\mathrm{Z}}\hat{\mathrm{a}}\hat{\mathrm{Z}}^{-1})(\hat{\mathrm{Z}}\hat{\mathrm{b}}\hat{\mathrm{Z}}^{-1}) \tag{3.24}$$

it follows directly that

$$0 = \hat{\mathrm{Z}}(\boldsymbol{\nabla}\hat{\mathrm{k}} - \hat{\mathrm{k}}\boldsymbol{\nabla})\hat{\mathrm{Z}}^{-1} = \boldsymbol{\nabla}_{\mathrm{h}}\hat{\mathrm{D}} - \hat{\mathrm{D}}\boldsymbol{\nabla}_{\mathrm{h}} \tag{3.25}$$

which applied to ϕ gives the irrotational property (3.19), and further, since $\boldsymbol{\nabla}^2 + \hat{\mathrm{k}}^2 = 0$ by definition,

$$0 = \hat{\mathrm{Z}}(\boldsymbol{\nabla}^2 + \hat{\mathrm{k}}^2)\hat{\mathrm{Z}}^{-1} = \boldsymbol{\nabla}_{\mathrm{h}}^2 + \hat{\mathrm{D}}^2, \tag{3.26}$$

which reproduces incompressibility (3.20). Evaluated order by order in the slope expansion, these imply

$$\sum_{m=0}^{n} [(\boldsymbol{\nabla}_{\mathrm{h}})_m w_{n-m} - \hat{\mathrm{D}}_m \boldsymbol{u}_{n-m}] = 0 \tag{3.27}$$

and

$$\sum_{m=0}^{n} [(\boldsymbol{\nabla}_{\mathrm{h}})_m \cdot \boldsymbol{u}_{n-m} + \hat{\mathrm{D}}_m w_{n-m}] = 0; \tag{3.28}$$

consequently in any finite truncation these properties are true only for a mixture of orders in the operators and velocity components. For example at consistent first order, the residual rotation is

$$\begin{aligned}(\boldsymbol{\nabla}_{\mathrm{h}})_{(1)} w_{(1)} - \hat{\mathrm{D}}_{(1)} \boldsymbol{u}_{(1)} &= (\boldsymbol{\nabla}_{\mathrm{h}})_1 w_1 - \hat{\mathrm{D}}_1 \boldsymbol{u}_1 \\ &= [\hat{\mathrm{D}}_1, (\boldsymbol{\nabla}\zeta)\hat{\mathrm{k}}]\phi,\end{aligned} \tag{3.29}$$

a quantity which can be shown to be small to third order in slope.

Is there a physical meaning to the breakdown of these properties in various truncated orders? Even with the approximate Hamiltonian one could in principle recover from ϕ the 'true'velocity, which has no such defects. On the other hand, the system also exhibits 'effective' velocities, in the form of vertical surface elevation rate and and horizontal advection rates for wave groups of infinitesimal group speed. These will turn out to be truncated velocities, whose failure to be precisely irrotational and incompressible will have consequences. Now, the horizontal gradients of these velocities are directly observable, while the vertical gradients are noticeable only indirectly, through their role in the formation of the perturbation equations. Here, the failure of quantities like (3.29) to vanish can be tied to the appearance of extra non-physical terms in the perturbation equations, as will be seen.

The explicit terms $\hat{\mathrm{D}}_n$ in (3.17) above can be inserted into the expression (2.15) for the normal flux operator $\hat{\mathrm{K}}$ and with some manipulation the Hamiltonian series exhibited through second nonlinear order as

$$H_0 = \frac{1}{2}\int [\phi\hat{\mathrm{k}}\phi + g\zeta^2]\,\mathrm{d}\boldsymbol{x}, \tag{3.30a}$$

$$H_1 = \frac{1}{2}\int \zeta[(\boldsymbol{\nabla}\phi)^2 - (\hat{\mathrm{k}}\phi)^2]\,\mathrm{d}\boldsymbol{x}, \tag{3.30b}$$

$$H_2 = -\frac{1}{4}\int (\hat{\mathrm{k}}\phi)\,[\zeta,[\zeta,\hat{\mathrm{k}}]]\,(\hat{\mathrm{k}}\phi)\,\mathrm{d}\boldsymbol{x}. \tag{3.30c}$$

The capillary potential energy has been omitted here for simplicity. The field equations through this order are

$$\frac{\partial\zeta}{\partial t} = w_0 + (w_1 - \boldsymbol{u}_0\cdot\boldsymbol{\nabla}\zeta) + (w_2 - \boldsymbol{u}_1\cdot\boldsymbol{\nabla}\zeta) \tag{3.31a}$$

and

$$\frac{\partial\phi}{\partial t} = -g\zeta - \tfrac{1}{2}[(\boldsymbol{\nabla}\phi)^2 - w_0^2] + (w_0 w_1), \tag{3.31b}$$

with the right-hand sides organized in ascending order – see (2.17), (2.18).

4. First-order perturbations

The equations for small perturbations ζ', ϕ' of an arbitrary solution of the exact field equations take a simple form when expressed in terms of the combination

$$\phi'' = \phi' - \zeta' w, \tag{4.1}$$

which can be read as the value of the perturbed potential on the unperturbed surface (Henyey *et al.* 1988). With

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} \tag{4.2}$$

denoting the substantial time derivative on the surface, these equations are

$$\frac{\mathrm{d}\zeta'}{\mathrm{d}t} + \zeta'\boldsymbol{\nabla}\cdot\boldsymbol{u} = \hat{\mathrm{K}}\phi'' \tag{4.3}$$

(see (2.11)–(2.13)) and

$$\frac{\mathrm{d}\phi''}{\mathrm{d}t} + g_*\zeta' = 0, \tag{4.5}$$

where g_* is the effective vertical component of gravity on the accelerating surface,

$$g_* = g + \frac{\mathrm{d}w}{\mathrm{d}t}. \tag{4.6}$$

One of the more rigorous requirements on an approximate dynamical system is that it reproduce the behaviour of short waves on long waves of finite amplitude. In the short-wave limit the equations (4.1)–(4.5) lead to the usual horizontal propagation and advection of phase and group properties along with such changes of amplitude as conserve action, according to eikonal equations in which the underlying long waves are represented entirely by the parameters $\boldsymbol{u}$, g_*, and slope $\boldsymbol{\nabla}\zeta$.

The truncated Hamiltonian provides only a set of equations describing the vertical displacement of an abstract surface; the velocity components $\boldsymbol{u}$, w are present parametrically in the equations, but in what sense are they present 'physically'? Actually the exact surface equations are equally abstract and the most direct evidence for horizontal velocity is the advection of 'test particles' in the form of eikonal wave packets of very short wavelength through the appearance of $\boldsymbol{u}$ in the substantial derivative $\mathrm{d}/\mathrm{d}t$. A first-order perturbation analysis of the truncated equations may similarly be a useful diagnostic both of the basic physical approximations and of potential failures in the modelling of long-wave interactions with short waves. Notice however that the modified potential ϕ'' in (4.1), the time derivative $\mathrm{d}/\mathrm{d}t$ in (4.2) and the effective acceleration g_* in (4.6) all contain velocity terms affected by the truncation approximation. For this reason, the approximate forms of (4.1)–(4.6) require some care in their derivation. It is best to start with the perturbation of the exact field equations and review the steps leading to the preferred form above, since these steps are also subject to approximation. The starting form for height perturbation is

$$\frac{\mathrm{d}\zeta'}{\mathrm{d}t}+\boldsymbol{u}'\cdot\nabla\zeta = w', \tag{4.7}$$

where primes denote perturbed values on the perturbed surface $\zeta+\zeta'$. From (3.15) we have

$$w' = (\hat{\mathrm{D}}\phi)' = \mathrm{D}\phi'+\zeta'\,\hat{\mathrm{D}}^2\phi-\hat{\mathrm{D}}\zeta'\,\hat{\mathrm{D}}\phi$$

or

$$w'' \equiv w'-\zeta'\,\hat{\mathrm{D}}w = \hat{\mathrm{D}}\phi'', \tag{4.8}$$

where double primes are values inferred on the unperturbed surface, while with the aid of (2.7) and (3.19) we have

$$\boldsymbol{u}'' \equiv \boldsymbol{u}'-\zeta'\,\hat{\mathrm{D}}\boldsymbol{u} = \nabla_{\mathrm{h}}\phi''. \tag{4.9}$$

These relations are linear in the terms $\hat{\mathrm{D}}$, $\nabla\zeta$ subject to the series approximation; substitution into (4.7) gives after simple rearrangement

$$\frac{\mathrm{d}\zeta'}{\mathrm{d}t}+\zeta'[\nabla\zeta\cdot\hat{\mathrm{D}}\boldsymbol{u}-\hat{\mathrm{D}}w] = [\hat{\mathrm{D}}-\nabla\zeta\cdot\nabla_{\mathrm{h}}]\phi'', \tag{4.10}$$

and linear substitution of (3.19), (3.20) identifies the first bracketed term as $\nabla\cdot\boldsymbol{u}$ while the second is $\hat{\mathrm{K}}$ by definition (2.11). This allows us to infer that the perturbation equation for ζ in the approximate system is simply the truncated version of (4.3). For the Hamiltonian of order $(n+1)$ in slope, the equation for ζ is also order $(n+1)$, while the equation for ζ' is order $(n+1)$ for those terms accompanying variations of ϕ, order (n) for those terms arising from surface variations ζ', and order (n) for $\mathrm{d}/\mathrm{d}t = \partial/\partial t+\boldsymbol{u}\cdot\nabla$. If we consistently define

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(n)} \equiv \frac{\partial}{\partial t}+\boldsymbol{u}_{(n)}\cdot\nabla \tag{4.11}$$

and

$$\phi''_0 \equiv \phi', \quad \phi''_n \equiv -\zeta' w_{n-1}, \tag{4.12}$$

then the truncated form of (4.3) is just

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(n)}\zeta'+\zeta'\nabla\cdot\boldsymbol{u}_{(n)} = \sum_{m+m'\leqslant n+1}\hat{\mathrm{K}}_m\,\phi''_{m'}. \tag{4.13}$$

Here, the series for $\hat{\mathrm{K}}$ is

$$\left.\begin{aligned} \hat{\mathrm{K}}_0 &= \hat{\mathrm{D}}_0 = \hat{\mathrm{k}}, \\ \hat{\mathrm{K}}_1 &= \hat{\mathrm{D}}_1 - \nabla\zeta\cdot\nabla, \\ \hat{\mathrm{K}}_m &= \hat{\mathrm{D}}_m + (\nabla\zeta)^2\,\hat{\mathrm{D}}_{m-2}, \quad m \geqslant 2. \end{aligned}\right\} \tag{4.14}$$

In the eikonal limit, as the wavenumber k' characteristic of the perturbation fields becomes large, the leading term on the right-hand side of (4.13) predominates,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(n)}\zeta' + \zeta'\nabla\cdot\boldsymbol{u}_{(n)} = \hat{\mathrm{k}}(\phi' - \zeta' w_{(n)}) + \epsilon_2 + \epsilon_3 \ldots \tag{4.15}$$

which suggests that $\phi' - \zeta' w_{(n)}$ is the preferred form of the modified potential for evaluating the approximations. To see this, write

$$[\zeta, \hat{\mathrm{k}}]f = \sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}\,\bar{\zeta}(\boldsymbol{k})\bar{f}(\boldsymbol{k}')\,[k' - |\boldsymbol{k}'+\boldsymbol{k}|] \tag{4.16}$$

and use
$$k' - |\boldsymbol{k}'+\boldsymbol{k}| = -\boldsymbol{k}\cdot\frac{\boldsymbol{k}'}{k'} + \frac{1}{2k'}\left[k^2 - \left(\boldsymbol{k}\cdot\frac{\boldsymbol{k}'}{k'}\right)^2\right] + \ldots$$

(which converges rapidly for $k' \gg k$) along with $\mathrm{i}\boldsymbol{k} \leftrightarrow \nabla$ to get

$$[\zeta, \hat{\mathrm{k}}]f = \nabla\zeta\cdot\nabla\hat{\mathrm{k}}^{-1}f + O(\nabla^2\zeta)\,\hat{\mathrm{k}}^{-1}f. \tag{4.17}$$

Then for $f = \hat{\mathrm{k}}\phi''$ we have

$$\hat{\mathrm{D}}_1\phi'' = \nabla\zeta\cdot\nabla\phi'' + O(\nabla^2\zeta)\,\phi''$$

or
$$\hat{\mathrm{K}}_1\phi'' = (\hat{\mathrm{D}}_1 - \nabla\zeta\cdot\nabla)\,\phi'' = 0 + O(\nabla^2\zeta)\,\phi''. \tag{4.18}$$

Consequently
$$\epsilon_2 = \hat{\mathrm{K}}_1(\phi' - \zeta' w_{(n-1)})$$

is small compared with the leading term, as is

$$\begin{aligned} \epsilon_3 &= \hat{\mathrm{K}}_2(\phi' - \zeta' w_{(n-2)}), \quad \hat{\mathrm{K}}_2 = -\tfrac{1}{2}\hat{\mathrm{k}}[\zeta, [\zeta, \hat{\mathrm{k}}]]\,\hat{\mathrm{k}} \\ &\approx \tfrac{1}{2}\hat{\mathrm{k}}[(\nabla\zeta)^2 - (\nabla\zeta\cdot\nabla\hat{\mathrm{k}}^{-1})^2]\,(\phi' - \zeta' w_{(n-2)}) \end{aligned} \tag{4.19}$$

and so on.

The corresponding approximation to (4.5) for ϕ'' is more complicated, first because the quantities $\mathrm{d}\phi''/\mathrm{d}t$ and g_* are products of series expansions, and second because in the exact perturbation of the potential equation,

$$\begin{aligned} 0 &= \frac{\mathrm{d}\phi'}{\mathrm{d}t} + \boldsymbol{u}'\cdot(\nabla\phi - \boldsymbol{u}) + g\zeta' - ww' \\ &= \frac{\mathrm{d}\phi''}{\mathrm{d}t} + \left(g + \frac{\mathrm{d}w}{\mathrm{d}t}\right)\zeta' + w\left(\frac{\mathrm{d}\zeta'}{\mathrm{d}t} + \boldsymbol{u}'\cdot\nabla\zeta - w'\right), \end{aligned} \tag{4.20}$$

the term multiplying w vanishes only to a particular order – infinite for the exact equations, and through $(n+1)$ for the approximate equation (4.13) – so that the sum above of mixed-order products up through order (n) generates leftover terms. While the truncated equation for ζ' is simple enough, no corresponding simple version of (4.20) is obvious for all finite truncation orders. Consequently, the appropriate terms must be assembled piecemeal from the truncated products. At leading nonlinear order this can be done almost trivially, without resort to the artillery accumulated in this section. Terms proliferate so quickly at next order however that the orderly identification provided by the foregoing equations is helpful, if not indispensable.

5. Leading nonlinear order

The equation for height perturbations at first nonlinear order reads

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \zeta' + \zeta' \nabla \cdot \boldsymbol{u}_0 = \hat{\mathrm{k}} \phi''_{(1)} + \hat{\mathrm{K}}_1 \phi''_0, \tag{5.1}$$

where again
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 = \frac{\partial}{\partial t} + \boldsymbol{u}_0 \cdot \nabla = \frac{\partial}{\partial t} + \nabla\phi \cdot \nabla, \tag{5.2}$$

and
$$\phi''_{(1)} = \phi''_0 - \zeta' w_0 = \phi' - \zeta' \hat{\mathrm{k}} \phi. \tag{5.3}$$

According to the discussion in §4 the term $\hat{\mathrm{K}}_1 \phi''_0$ will become negligible in the limit of large perturbation wavenumber k'. The companion equation starts with the perturbed form of (3.31 b) through first nonlinear order,

$$\frac{\partial \phi'}{\partial t} + g\zeta' + \nabla\phi \cdot \nabla\phi' - w_0 w'_0 = 0, \tag{5.4}$$

and becomes, with the aid of (5.2), (5.3),

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \phi''_{(1)} + g_{(1)} \zeta' + w_0 \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \zeta' - w'_0\right] = 0, \tag{5.5}$$

where
$$g_{(1)} = g + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 w_0. \tag{5.6}$$

Through (5.1), or equivalently, (4.7), the term in (5.5) enclosed in square brackets can be recognized as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \zeta' - w'_0 = [\zeta', \hat{\mathrm{k}}]\, w_0 \tag{5.7}$$

so that
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \phi''_{(1)} + g_{(1)} \zeta' + w_0 [\zeta', \hat{\mathrm{k}}]\, w_0 = 0. \tag{5.8}$$

The final term above is not present in the exact equations and arises only in the truncation approximation at lowest order. It can be written as

$$w_0[\zeta', \hat{\mathrm{k}}]\, w_0 = -w_0^2 \hat{\mathrm{k}} \zeta' + \zeta'(w_0 \hat{\mathrm{k}} w_0) + w_0[w_0, \hat{\mathrm{k}}]\, \zeta', \tag{5.9}$$

in which the first term dominates as $k' \to \infty$, because according to the argument of §4 the third term approaches

$$[w_0, \hat{\mathrm{k}}]\, \zeta' \doteq \nabla w_0 \cdot (\hat{\mathrm{k}}^{-1} \nabla \zeta'), \tag{5.10}$$

which is of order k'/k smaller than the first term, as is the second term (k denoting the wavenumber scale of the unperturbed velocity w_0).

To this order in short wavelength the perturbation equations are

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \zeta' + \zeta' \nabla \cdot \boldsymbol{u}_0 - \hat{\mathrm{k}} \phi''_{(1)} \doteq 0 \tag{5.11a}$$

and
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 \phi''_{(1)} + (g_{(1)} - w_0^2 \hat{\mathrm{k}})\, \zeta' \doteq 0. \tag{5.11b}$$

Apart from the spurious term accompanying $g_{(1)}$ these resemble the exact equations, which are those of linearized waves in a coordinate system of horizontal velocity $\boldsymbol{u}_0 = \nabla\phi$ and vertical acceleration $(\mathrm{d}/\mathrm{d}t)_0 w_0$. It is worth remarking that the first nonlinear term in the Hamiltonian is entirely responsible for the advection of waves by currents associated with other waves, through the terms $\boldsymbol{u}_0\cdot\nabla$ in the pair of equations above. The displacement of short waves riding on long waves is fully comparable with the long-wave height, which can amount to many wavelengths of the short waves. Additional nonlinear orders do not materially change the advective behaviour, beyond improving the approximation to $\boldsymbol{u}$. There is no immediate difficulty involving large products of k' and wave height, as there would be in the usual Stokes expansion of the solutions. Large ratios of wavelength present a more indirect problem at lowest order through the action of the extra term.

Unfortunately the extra term is always negative, so that wavenumbers k' exist that drive the effective local gravity negative,

$$\tilde{g} = g_{(1)} - w_0^2 k', \tag{5.12}$$

for any finite surface velocity. The intrinsic frequency ω', given by

$$\omega'^2 = \tilde{g}k', \tag{5.13}$$

is imaginary at these wavenumbers, and the associated perturbation modes are therefore exponentially unstable. Any finite, band-limited simulation of these equations will remain stable provided that

$$|w_0| < c_{\mathrm{min}}, \tag{5.14}$$

where c_{min} is the phase speed of the shortest wave present,

$$c_{\mathrm{min}}^2 = g/k_{\mathrm{max}}. \tag{5.15}$$

At finite slope ϵ the vertical velocity is of order $w_0 \sim \epsilon c_{\mathrm{max}}$, where c_{max} is the phase speed of the longest wave present, so that the nonlinearity must be limited by

$$\epsilon < c_{\mathrm{min}}/c_{\mathrm{max}} = (k_{\mathrm{min}}/k_{\mathrm{max}})^{\frac{1}{2}} \tag{5.16}$$

to preserve stability. The presence of an added capillary term $\tau\hat{\mathrm{k}}^2$ (approximately; see (2.9)) can stabilize the equations at all wavenumbers. The necessary condition is easily shown to be

$$|w_0| < c_{\mathrm{min}} = (4\tau g)^{\frac{1}{4}}. \tag{5.17}$$

6. Second nonlinear order

Two equivalent forms of the truncated perturbation equation for height are useful at second nonlinear order. The first form is just (4.13) made explicit,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)} \zeta' + \zeta'\nabla\cdot\boldsymbol{u}_{(1)} - \hat{\mathrm{k}}\phi''_{(2)} = \hat{\mathrm{K}}_2\phi' - \hat{\mathrm{K}}_1(\zeta' w_0) \doteq 0, \tag{6.1}$$

coupling ζ' to the source term in $\phi''_{(2)}$, with terms on the right-hand side that become negligible in the short-wavelength limit. The second form,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)} \zeta' + \boldsymbol{u}'_{(1)}\cdot\nabla\zeta - w'_{(2)} = 0, \tag{6.2}$$

will be useful in deriving the equation for ϕ'' below. The substantial derivative carries the advective velocity to one order higher than before,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)} = \frac{\partial}{\partial t} + (\boldsymbol{\nabla}\phi - w_0\boldsymbol{\nabla}\zeta)\cdot\boldsymbol{\nabla}, \tag{6.3}$$

now including a slope correction to $\boldsymbol{\nabla}\phi$.

The exact equation (4.20) for ϕ'' leaves so much debris in the form of mixed-order products when truncated directly that an *ad hoc* approach based on the original approximate equation (3.31 *b*) for ϕ is preferable. That equation and its perturbation are

$$\frac{\partial\phi}{\partial t} + g\zeta + \tfrac{1}{2}[(\boldsymbol{\nabla}\phi)^2 - w_{(1)}^2] + w_1^2 = 0, \tag{6.4}$$

and

$$\frac{\partial\phi'}{\partial t} + g\zeta' + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi' - w_{(1)}w'_{(1)} + w_1 w'_1 = 0. \tag{6.5}$$

The term $\frac{1}{2}w_1^2$ has been added and subtracted above to simplify the subsequent collection of terms at inclusive order (1), which is the order necessary to make the resulting equation compatible with (6.1). Adding and subtracting $w_0\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\phi'$ yields

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)}\phi' + g\zeta' + w_{(1)}[\boldsymbol{u}'_0\cdot\boldsymbol{\nabla}\zeta - w'_{(1)}] + w_1(w'_1 - \boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\phi') = 0, \tag{6.6}$$

while adding and subtracting $(\mathrm{d}/\mathrm{d}t)_{(1)}\zeta' w_{(1)}$ gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)}\phi''_{(1)} + \left[g + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)} w_{(1)}\right]\zeta' + w_{(1)}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)}\zeta' + \boldsymbol{u}'_0\cdot\boldsymbol{\nabla}\zeta - w'_{(1)}\right] + w_1(w'_1 - \boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\phi') = 0. \tag{6.7}$$

This is close to the desired form; the third and fourth terms, which would vanish at infinite order, are the truncation residue. To evaluate the third term, note that the bracketed part resembles (6.2) except that the highest-order terms are missing. Subtracting $w_{(1)}$ times (6.2) then yields

$$-w_{(1)}(\boldsymbol{u}'_1\cdot\boldsymbol{\nabla}\zeta - w'_2) = -w_{(1)}[\zeta'\boldsymbol{\nabla}\cdot\boldsymbol{u}_1 - \hat{\mathrm{K}}_2\phi' + \hat{\mathrm{K}}_1(\zeta' w_0) + \hat{\mathrm{k}}\zeta' w_1]; \tag{6.8}$$

see (4.13). The remaining term can be directly evaluated via (4.8) and (4.14) as

$$w_1(w'_1 - \boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\phi') = w_1(\hat{\mathrm{K}}_1\phi' + [\zeta', \hat{\mathrm{k}}]w_0). \tag{6.9}$$

Two terms in the expressions above consist of ζ' multiplied by small scalar (not operator) quantities, which can be added directly to the effective gravitation:

$$g_{(2)} \equiv g + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)} w_{(1)} + w_{(1)}\boldsymbol{\nabla}\cdot(w_0\boldsymbol{\nabla}\zeta) + w_1\hat{\mathrm{k}}w_0. \tag{6.10}$$

Two other terms involving ζ' standing to the right of $\hat{\mathrm{k}}$ are important, because they dominate at $k'\to\infty$:

$$-(w_{(1)}\hat{\mathrm{k}}\zeta' w_1 + w_1\hat{\mathrm{k}}\zeta' w_0) = -(w_0 + w_{(1)})w_1\hat{\mathrm{k}}\zeta' + \{w_{(1)}[w_1, \hat{\mathrm{k}}] + w_1[w_0, \hat{\mathrm{k}}]\}\zeta'. \tag{6.11}$$

Here they have been separated into terms proportional to k' and independent of k', according to the asymptotic behaviour of the commutator expressions. Remaining terms are small absolutely ($\hat{\mathrm{K}}_2$) or in the short-wavelength limit ($\hat{\mathrm{K}}_1$). Accordingly, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{(1)}\phi''_{(1)}+[g_{(2)}+(w_0^2-w_{(1)}^2)\,\hat{\mathrm{k}}]\,\zeta' = R_{(2)} = 0, \tag{6.12}$$

with
$$R_{(2)} = w_{(1)}[\hat{\mathrm{K}}_1(\zeta' w_0)-\hat{\mathrm{K}}_2\,\phi'-[w_1,\hat{\mathrm{k}}]\,\zeta']-w_1[\hat{\mathrm{K}}_1\,\phi'+[w_0,\hat{\mathrm{k}}]\,\zeta']. \tag{6.13}$$

Equations (6.12) and (6.13) are improvements over their counterparts at lowest order in two respects. The ordinary parameters $\boldsymbol{u}$ and $\mathrm{d}w/\mathrm{d}t$ are better approximated, and the anomalous term accompanying g is smaller by a factor of the unperturbed wave slope ϵ. Exponential instability, though still present, requires higher wavenumber or higher slope, such that

$$(w_0^2-w_{(1)}^2)\frac{k'}{g} = \frac{w_0^2-w_{(1)}^2}{c_{\mathrm{min}}^2} < -1. \tag{6.14}$$

Such instability is confined to regions where $|w_0| < |w_{(1)}|$ and because $|w_1| \sim \epsilon|w_0|$, the equations at this order can stably tolerate slopes not exceeding

$$\epsilon \sim (c_{\mathrm{min}}/c_{\mathrm{max}})^{\frac{2}{3}} = (k_{\mathrm{min}}/k_{\mathrm{max}})^{\frac{1}{3}}. \tag{6.15}$$

This relation can be made more precise only for specific cases. For example a steady progressive wave described by

$$\zeta \doteq \epsilon k^{-1}\cos\theta, \quad \phi \doteq \epsilon c k^{-1}\sin\theta, \quad \theta = k(x-ct), \tag{6.16}$$

in the limit of small slope ϵ, has for the quantities above the following values:

$$w_0 = \hat{k}\phi = \epsilon c\sin\theta, \tag{6.17}$$

$$w_1 = [\zeta,\hat{k}]\,w_0 = -\epsilon^2 c\sin\theta\cos\theta, \tag{6.18}$$

and
$$w_{(1)}^2-w_0^2 = -2\epsilon^3 c^2\sin^2\theta\cos\theta. \tag{6.19}$$

Consequently the stability parameter can be written

$$\frac{w_{(1)}^2-w_0^2}{c_{\mathrm{min}}^2} \doteq \epsilon^3\frac{k_{\mathrm{max}}}{k}F(\theta), \tag{6.20}$$

in which
$$F(\theta) = -2\sin^2\theta\cos\theta \tag{6.21}$$

reaches its maximum value of 0.770 at positions just behind and ahead of the wave trough, $\theta = 180° \pm 54.7°$. These are the locations at which the exponential instability first appears when the quantity (6.20) just exceeds unity. The limiting stable slope is plotted in figure 1 as a function of the wavenumber ratio k_{max}/k. The curve is accurate only for small values of slope, but it is qualitatively informative at larger slopes, particularly in the comparison with the stability bound for the simpler Hamiltonian of §5.

7. Summary

The truncation of the surface-wave Hamiltonian to finite order in slope induces approximations both in the dynamics and in the surface kinematics. The relations among potential, elevation rate, and horizontal velocity are truncated at finite order, causing higher-order imperfections to appear in the equivalent expressions for incompressibility and irrotationality at the surface. The equations of motion are nevertheless quite fluid-like even at first or second nonlinear order, particularly in the fidelity with which they describe the transport of small waves by larger waves.

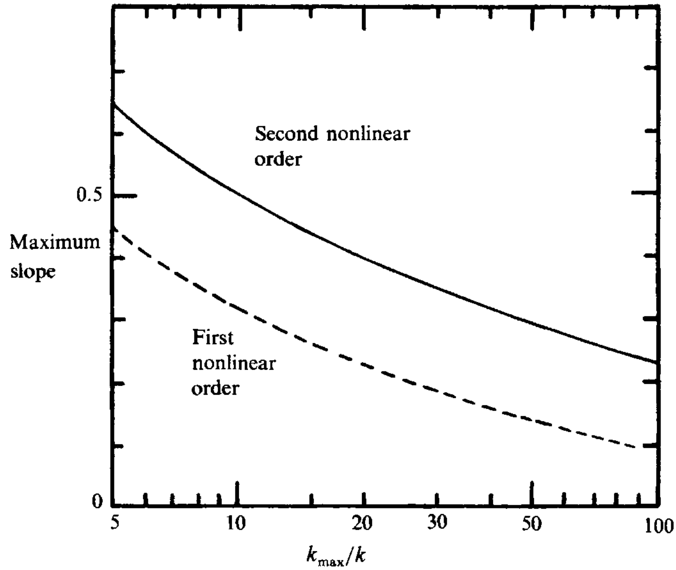

FIGURE 1. Slopes for progressive 'Stokes' waves above which modes at wavenumber k_{max} become exponentially unstable.

However, the perturbation analysis that gives this reassuring result also reveals instabilities which accompany any finite-amplitude motion for the two lowest-order Hamiltonians analysed. These instabilities affect modes whose phase speed is less than a local critical value; at first nonlinear order the critical velocity is just the surface elevation rate, while at next order it is the harmonic mean of this quantity and its first-order correction, so that the unstable wavenumber is increased by a factor of the inverse wave slope. The addition of the capillary force can stabilize the flow at all wavenumbers when the critical velocity parameter is kept below the minimum phase speed.

So far unanswered is the question of whether this kind of instability vanishes when terms beyond second nonlinear order are retained in the Hamiltonian, or whether the instability merely continues to recede to higher slopes and wavenumbers.

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